Homework 1: Due September 17 (11:55 p.m.)

Instructions

- **Answer each question on a separate page.**
- **Honors questions are optional. They will not count towards your grade in the course. However you are encouraged to submit your solutions on these problems to receive feedback on your attempts. Our estimation of the difficulty level of these problems is expressed through an indicative number of stars ($'*' =$ easiest) to ($'*****' =$ hardest).**
- **You must enter the names of your collaborators or other sources as a response to Question 0. Do NOT leave this blank; if you worked on the homework entirely on your own, please write "None" here. Even though collaborations in groups of up to 3 people are encouraged, you are required to write your own solution.**

Question 0: List all your collaborators and sources: ($-\infty$ points if left blank)

Question 1: (1+6+3=10 points)

Prove the following equality using induction on n:[1]

$$(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n \text{ for all } n \in \mathbb{N}\ . \tag{1}$$

1. Check the base case ($n = 1$).
2. Prove the inductive step.
3. Using Eq. (1), evaluate the following sum:

$$3^n + 2\cdot 3^{n-1} + 2^2\cdot 3^{n-2} + \cdots + 2^n = ???$$

Question 2: (1+2+2+3+2=10 points)

$f = \mathcal{O}(g)$ is defined for functions f and g (both from $\mathbb{N}$ to $\mathbb{N}$) to mean that there exist positive constants n_0 and C such that:

$$f(n) \le C \cdot g(n) \text{ for all } n \ge n_0.$$

For each of the following statements either prove the statement if it is true or otherwise provide a counterexample and justify why your counterexample is indeed a counterexample:

1. If $f = \mathcal{O}(g)$ then $g = \mathcal{O}(f)$.
2. If $f = \mathcal{O}(g)$ and $g = \mathcal{O}(h)$ then $f = \mathcal{O}(h)$.
3. If $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$ and $\forall n, f(n) > g(n)$ then $f - g = \mathcal{O}(1)$.
4. If $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$ then $\frac{f}{g} = \mathcal{O}(1)$.
5. If $f = \mathcal{O}(g)$ and $h = \mathcal{O}(g)$ then $f = \mathcal{O}(h)$.

[1] Recall that $\mathbb{N}$ denotes the set $\{1, 2, \ldots\}$.

Question 3: (5 points)

Rank the following functions by order of growth (You need not prove the correctness of your ranking). That is, find an order f_a, f_b, $f_c \dots f_e$ so that $f_a = \mathcal{O}(f_b)$, $f_b = \mathcal{O}(f_c)$, and so on:

a) $2^{\log_3 n}$

b) $\sqrt{n} \log_2 n$

c) 2^n

d) $n!$ where $n! = 1 \cdot 2 \cdot 3 \dots (n-1) \cdot n$ so for example $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

e) n^2

Question 4: (5 points)

By f_n we denote the n-th Tribonacci number. Tribonacci numbers are defined by $f_1 = f_2 = 0$, $f_3 = 1$, and $f_n = f_{n-1} + f_{n-2} + f_{n-3}$ for $n \geq 4$. Thus the Tribonacci sequence goes as:

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, \dots$$

Prove (by induction on n) that $f_n > 3n$ for all $n > 9$.

Question 5: (2+3+5=10 points)

Let $A[1, 2, \dots, n]$ be an array of n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called a *transposition* of A. Answer the following questions:

1. List all the transpositions of the array $(7, 5, 2, 6, 11)$
2. Which arrays with distinct elements from the set $\{1, 2, \dots, n\}$ have the smallest and the largest number of transpositions and why? State the expressions exactly in terms of n.
3. Give an algorithm that determines the number of transpositions in an array consisting of n numbers in $\Theta(n \log n)$ worst-case time. Also prove the correctness and run time bounds for your algorithm. (Hint: Modify merge sort.)

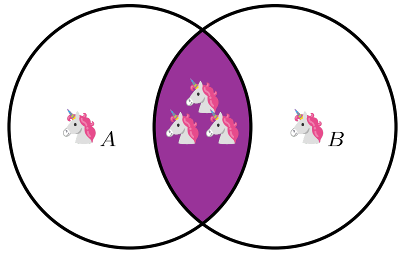

Figure 1: Horses A and B, with all the rest of the horses lying in the violet region common to both the sets.

Question 6: (5 points)

Find a flaw in the following "proof by induction" (see Figure 1 for an illustration). In particular, state why the inductive step is incorrect:

Claim: For all $n \in \mathbb{N}$, and any set of n horses, all horses in the set have the same color.

1. Base Case ($n = 1$): If there is just one horse in the set, obviously all horses have the same color.
2. Inductive Step: Suppose the induction hypothesis holds for all $1, 2, \ldots, n$. Our goal is to prove the statement for sets of $n + 1$ horses. So take any such set. Now exclude one horse, call this horse A, and look at the set of n remaining horses. By the induction hypothesis, they all have the same color. Now exclude a different horse, call it B, and look at the set of n remaining horses, which includes horse A. Then, all horses in this set must also have the same color. This implies that A and B also have the same color. Hence, we obtain that all $n+1$ horses in our set have the same color, "proving" the claim.

Extra Practice (Optional, 0 points)

Prove by induction: A convex n-gon has $n(n-3)/2$ diagonals. A convex n-gon is a shape with n angles such that each interior angle is less than or equal to 180 degrees. A diagonal is a line segment connecting any two non-adjacent vertices. For example, a triangle is a "3-gon" and a pentagon is a "5-gon".

Honors Question (Optional, 0 points)

(**) Intuitively, $f = O(g)$ means that f is upper bounded by g multiplied by some positive constant. What if we wanted to also disregard multiplicative logarithmic factors (in addition to constant factors) when comparing two functions? For example, we would like to write $\log_2 n + n \log_2^2 n = \tilde{O}(n)$ for some appropriately defined notation $\tilde{O}(\cdot)$. Formally define the notation $f = \tilde{O}(g)$ that achieves this. Comment on what properties $\tilde{O}(\cdot)$ has (look at Question 2 for inspiration).